This paper shows the derivation of a closed form solution for the ultimate bending capacity of any symmetrical reinforced concrete section subject to axial load, with any number of layers of passive or prestressed reinforcement.

The analysis uses a rectangular compressive stress distribution in the concrete, as permitted by most design codes. Other assumptions are:

- Plane sections remain plane.
- Tensile stresses in the concrete are ignored.
- Bi-linear elastic/perfectly plastic stress/strain behaviour of the reinforcement in tension and compression.
- Specified strain at the compression face.

The analysis procedure is as follows:

1. Select a design ultimate axial load for the section, $\phi P_u$
2. Find the axial load for “balanced stress” conditions, according to the applicable design code.
3. Find $\phi$, and the ultimate axial force, $P_u$
4. Find the depth of the neutral axis for the force $P_u$
5. Find the strain, stress, and force in each reinforcement layer, and check that the total nett force on the section is equal to $P_u$
6. Find the total nett bending moment of the reinforcement and the concrete compression zone, about the centroid of the gross concrete section, $M_u$
7. The design ultimate bending moment is then $\phi M_u$
Finding the Level of the Neutral Axis

Consider a reinforced concrete section with:

- Applied axial load, $P$, (compression positive) and associated bending moment, $M$, such that the strain at the extreme compression face is equal to the specified ultimate compressive strain of the concrete, $\varepsilon_{cu}$.
- X axis on the section Neutral Axis, $y$ positive towards the compression face.
- Extreme compression face at $Y$ above the Neutral Axis.
- Uniform concrete compressive stress of a specified proportion of the design compressive strength, $\alpha f'_c$, extending over a specified proportion of the region from the compressive face to the Neutral Axis, $yY$

The level of the Neutral Axis, $Y$, is to be found, such that the total reaction force in the section, $N$, is equal to the nominal applied load, $P/\phi$ (see capacity reduction factor below).

For a rectangular section:

Width = $B$
Height of zone above the Neutral Axis = $Y$
Concrete stress = $\alpha f'_c$
Depth of concrete compression zone = $yY$

$$P = B.\gamma Y. \alpha f'_c$$

For a section with one layer of reinforcement:

Reinforcement area = $a_{st}$
$E_s$ = Young’s Modulus of the reinforcing steel
$f_{ys}$ = Steel yield stress
$\varepsilon_{ps}$ = Steel strain due to prestress
Depth below compression face = $d_{st}$
Steel strain = $\varepsilon_{cu}(1 - d_{st}/Y)$
Steel stress = $-f_{ys} < (\varepsilon_{cu}(1 - d_{st}/Y) - \varepsilon_{ps})E_s < f_{ys}$
for layers outside the concrete compression zone or:
$-f_{ys} - \alpha f'_c < (\varepsilon_{cu}(1 - d_{st}/Y) - \varepsilon_{ps})E_s - \alpha f'_c < f_{ys} - \alpha f'_c$
for layers inside the concrete compression zone.

For reinforcement layers outside the concrete compression zone:

For reinforcement strain exceeding tensile yield strain:

$$P = B.\gamma Y. \alpha f'_c - f_{ys} \cdot a_{st}$$
$$B.\gamma Y^2 \cdot \alpha f'_c - (f_{ys} \cdot a_{st} + P)Y = 0 \quad (1.1)$$

For reinforcement strain in the elastic range:

$$P = B.\gamma Y. \alpha f'_c + (\varepsilon_{cu}(1 - d_{st}/Y) - \varepsilon_{ps})E_s \cdot a_{st}$$
$$B.\gamma Y^2 \cdot \alpha f'_c + (\varepsilon_{cu}(Y - d_{st}) - \varepsilon_{ps})E_s \cdot a_{st} - PY = 0 \quad (1.2)$$

For reinforcement layers inside the concrete compression zone:

For reinforcement strain exceeding compressive yield strain:

$$P = B.\gamma Y. \alpha f'_c + (f_{ys} - \alpha f'_c) \cdot a_{st}$$
$$B.\gamma Y^2 \cdot \alpha f'_c + ((f_{ys} - \alpha f'_c) \cdot a_{st} - P)Y \quad (1.3)$$
For reinforcement strain in the elastic range:

\[ P = B.\gamma Y \cdot \alpha f'_c + ((\varepsilon_{cu} - \varepsilon_{ps})E_s - \alpha f'_c) \cdot a_{st} \]

\[ B.\gamma Y^2 \cdot \alpha f'_c + ((\varepsilon_{cu}(Y - d_{st}) - (\varepsilon_{ps}) Y)E_s - \alpha f'_c) \cdot a_{st} - PY = 0 \quad (1.4) \]

Substituting into \( aY^2 + bY + c = 0 \)

All cases: \( a = B.\gamma \cdot \alpha f'_c \) \( (1.5) \)

Equation 1.1: \( b = -(f_{ys} \cdot a_{st} + P) \) \( (1.6) \)
Equation 1.2: \( b = (\varepsilon_{cu} - \varepsilon_{ps})E_s \cdot a_{st} - P \) \( (1.7) \)
Equation 1.3: \( b = ((f_{ys} - \alpha f'_c) \cdot a_{st} - P) \) \( (1.8) \)
Equation 1.4: \( b = ((\varepsilon_{cu} - \varepsilon_{ps})E_s - \alpha f'_c) \cdot a_{st} - P \) \( (1.9) \)

Equations 1.1, 1.3: \( c = 0 \)
Equations 1.2, 1.4: \( c = -(\varepsilon_{cu} \cdot d_{st})E_s \cdot a_{st} \) \( (1.10) \)

For multiple reinforcement layers:
Sum b and c values, using equations 1.6 to 1.10

Non-rectangular concrete sections are handled by dividing the section into two layers, an upper layer of any shape, and a rectangular or trapezoidal lower layer, containing the base of the rectangular stress block.

For the top layer:

Depth of layer = \( d_t \)

Depth of layer centroid below compression face = \( d_{ct} \)

Area = \( A_t \)

Then:

For a rectangular layer of width B containing the base of the stress block:

\[ P = B.(\gamma Y - d_t) \cdot \alpha f'_c \]

Total of the two layers:

\[ P = (A_t + B.(\gamma Y - d_t)) \cdot \alpha f'_c \]
Finally for a trapezoidal lower layer the neutral axis equation becomes a cubic:
\[ aY^3 + bY^2 + cY + d = 0 \]

If:
\[ K = (B2 - B1)/D \]
\[ B = B1 \]

For a trapezoidal layer of top width B, containing the base of the stress block:

\[ P = (2B + K(yY - d))/2 (yY - d). \alpha f'_c \]
\[ P = (K/2 . y^2)Y^2 + y(B - K d)Y - B d + K d^2 /2). \alpha f'_c \]

Summing the two layers, and multiplying by Y:
\[ (K/2 . y^2)Y^3 + y(B - K d)Y^2 + (A - B d + K d^2 /2)Y). \alpha f'_c - PY = 0 \]

Hence the cubic coefficients for the most general case (including the coefficients for the reinforcement, previously derived) are:

All cases:
\[ a = (K/2 . y^2) \alpha f'_c \] (2.1)
\[ b = y(B - K dt)\alpha f'_c \] (2.2)
\[ c = \left( A - B dt + \frac{K}{2} dt^2 \right) \alpha f'_c - \sum P_{st} - P \] (2.3)

Where \( \sum P_{st} \) = Sum of reinforcement layer forces:

Equation 1.1: \( P_{st} = (fys \cdot a_{st}) \) (2.3.1)
Equation 1.2: \( P_{st} = (\varepsilon cu - \varepsilon ps)Es \cdot a_{st} \) (2.3.2)
Equation 1.3: \( P_{st} = fys - \alpha f'_c \cdot a_{st} \) (2.3.3)
Equation 1.4: \( P_{st} = ((\varepsilon cu - \varepsilon ps)Es - \alpha f'_c) \cdot a_{st} \) (2.3.4)

Equations 1.1, 1.3: \( d = 0 \)
Equations 1.2, 1.4: \( d = -(\varepsilon cu \cdot d_{st})E_s \cdot a_{st} \) (2.4)

**Capacity Reduction Factor, \( \phi \)**

In the Australian codes a capacity reduction factor, \( \phi \), is applied to the nominal axial capacity and bending capacity to derive the design values, and the value of \( \phi \) varies depending on the applied axial load, and the balance axial load, \( N_{ub} \).

\( N_{ub} \) is defined as the ultimate strength in compression when \( k_{u0} = 0.003/(0.003 + f_{sy}/E_s) \)
If \( N_u \geq N_{ub} \) Then \( \phi = 0.6 \)
If \( N_u < N_{ub} \) Then \( \phi = 0.6 + [(\phi_0 - 0.6) (1 - N_u / N_{ub})] \)
Where \( \phi_0 \) is the value of \( \phi \) for a section with zero axial load.

For the purposes of determining the value of \( \phi \) it is normal to assume that the load eccentricity, \( M/P \) is constant. In order to find the design bending capacity, \( \phi M_u \) of a section with an applied axial load \( N \):
Assume \( N = \phi N_u \)
Determine the value of \( N_{ub} \)
If \( 0 < N < N_{ub} \) then:
\[
\phi = 0.6 + [(\phi_0 - 0.6) (1 - N_u / N_{ub})] \\
\phi = 0.6 + [(\phi_0 - 0.6) (1 - \phi N_{ub})] \\
\phi^2 = 0.6 \phi + [(\phi_0 - 0.6) (\phi - N / N_{ub})] \\
\phi^2 - (0.6 \phi + [(\phi_0 - 0.6) (\phi - N / N_{ub})]) = 0 \\
\phi^2 - \phi_0 \phi + (N / N_{ub}) (\phi_0 - 0.6) = 0 \\
\]
If \( (N / N_{ub}) (\phi_0 - 0.6) = c \) then:
\[
\phi = (\phi_0 + (\phi_0^2 - 4c)^{0.5}) / 2 \\
\]
If \( 4c > \phi_0^2 \) Then \( N_u > N_{ub} \) and \( \phi = 0.6 \)

**Example 1:**
Rectangular section: 1000 mm wide x 300 mm deep, \( f'c = 40 \text{ MPa} \)
Reinforcement: Top, 5 No Y16, depth = 60 mm; Bottom, 5 No Y20, depth = 250 mm, \( E = 200 \text{ GPa} \)
Axial load = 500 kN

\[ \alpha = 0.85 \]
\[ \alpha f'c = 0.85 \times 40 = 34 \]
\[ \gamma = 0.85 - 0.007(40 - 28) = 0.766 \]
Reinforcement areas: top = 1005 mm\(^2\); bottom = 1571 mm\(^2\)
\[ k_{a0} = 0.003/(0.003 + 500/200000) = 0.5455 \]
Balance load, \( N_{ub} = 3070 \text{ kN} \)

\[ c = (N / N_{ub}) (\phi_0 - 0.6) = (500 / 3070)(0.8 - 0.6) = 0.03257 \] (3.1)
\[ \phi = (\phi_0 + (\phi_0^2 - 4c)^{0.5}) / 2 = (0.8 + (0.64 - 4 \times 0.03257)^{0.5}) / 2 = 0.7570 \] (3.2)

\[ N_u = 500000 / 0.7570 = 660525 \text{ N} \]
\[ a = 1000 \times 0.766 \times 34 = 26044 \] (1.5)
Assuming that both reinforcement layers are outside the concrete compressive zone, and that the top reinforcement is in the elastic range and the bottom reinforcement has yielded, apply Equation 1.1 to the bottom reinforcement, and 1.2 to the top reinforcement:

\[
b = (0.003 - 0) \times 200000 \times 1005 - (500 \times 1571) - 660525 = -842746
\]

\[
c = - (0.003 \times 60) \times 200000 \times 1005 = -3.619E07
\]

Solving the quadratic polynomial for \( Y \) yields \( Y = 56.81 \text{ mm} \)

Concrete Force = \( 1000 \times 56.81 \times 0.766 \times 34 / 1000 = 1480 \text{ kN} \)
Top reinforcement strain = \( 0.003 \times (1 - 60/56.81) = -1.68E-04 \)
Top reinforcement force = \( 1005 \times 200000 \times -1.68E-04 / 1000 = -34 \text{ kN} \)
Bottom reinforcement strain = \( 0.003 \times (1 - 250/56.81) = -1.02E-02 \) > yield strain
Bottom reinforcement force = \( 1571 \times 500 / 1000 = 785 \text{ kN} \)
\( N_u = 1480 - 34 - 785 = 661 \text{ kN} \)
\( \phi N_u = 661 \times 0.7570 = 500 \text{ kN} \quad \text{OK} \)

Taking moments about the centre of the section:
Concrete lever arm = \( 150 - (56.81 \times 0.766)/2 = 128.2 \text{ mm} \)
Concrete moment = \( 1480 \times 128.2 / 1000 = 190 \text{ kNm} \)
Reinforcement moment = \( (-34 \times (150 - 60) - 785 \times (150 - 250)) / 1000 = 75 \text{ kNm} \)
\( M_{u} = 265 \text{ kNm} \)
\( \phi M_u = 265 \times 0.7570 = 201 \text{ kNm} \)

**Example 2:**
**Circular section:** 600mm diameter, \( f'c = 32 \text{ MPa} \)
Reinforcement: 12 No. Y 20 bars, cover = 30 mm, \( E = 200 \text{ GPa} \)
Axial load = \( 2600 \text{ kN} \)

By dividing the cross section into layers it is found that the bottom of the concrete compression zone is between 300 and 350 below the compression fibre; therefore divide the circular section into two parts:

A semi-circle of radius 300 mm.
A trapezoidal layer: \( B1 = 600 \text{ mm}, B2 = 591.6 \text{ mm}, Dl = 50 \text{ mm} \).
\[ \alpha = 0.85 \]
\[ \alpha f'c = 0.85 \times 32 = 27.20 \text{ MPa} \]
\[ y = 0.85 - 0.007(32 - 28) = 0.822 \]
Reinforcement areas: top and bottom = 314.2 mm\(^2\); other layers = 628.3 mm\(^2\)
\[ k_{0} = 0.003/(0.003 + 500/200000) = 0.5455 \]
Balance load, \( N_{ub} = 3021 \text{ kN} \)
\[ c = (N / N_{ub})(\phi_0 - 0.6) = (2600 / 3021)(0.8 - 0.6) = 0.17213 \] \( (3.1) \)
\[ 4c > 0.64; N_{u} > N_{ub}; \phi = 0.6 \]
\[ N_{u} = 2600000 / 0.6 = 4333333 \text{ N} \]
\[ A = \pi 300^2 / 2 = 141372 \text{ mm}^2 \]
\[ K = (591.6-600)/50 = -0.1680 \]

The cubic coefficients are:

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Solving the cubic polynomial for \( Y \) yields \( Y = 374.7 \text{ mm} \)

Depth of stress block = 374.7 * 0.822 = 308.0

Concrete Force = 3954 kN
Total reinforcement force = 380 kN
\[ N_{u} = 3954 + 380 = 4334 \text{ kN} \]
\[ \phi N_{u} = 4334 \times 0.60 = 2600 \text{ kN} \quad \text{OK} \]

Taking moments about the centre of the section:
Concrete moment = 485 kNm
Reinforcement moment = 183 kNm
\[ M_{u} = 668 \text{ kNm} \]
\[ \phi M_{u} = 668 \times 0.60 = 400 \text{ kNm} \]

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