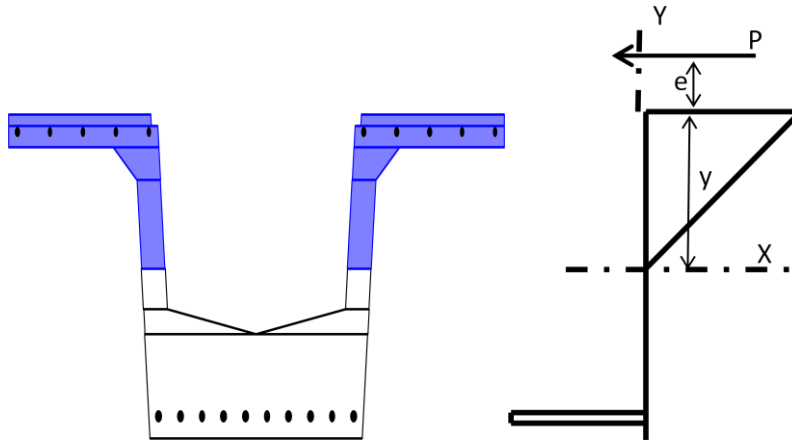


# Neutral Axis Depth for a Reinforced Concrete Section

## Under Eccentric Axial Load

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To determine the stresses in a reinforced concrete section we must first find the position of the neutral axis. The procedure below provides a closed form solution for the depth of the neutral axis in any symmetrical reinforced concrete section subject to eccentric axial load, with any number of layers of passive or prestressed reinforcement. A similar analysis is given in Reference (1) for a rectangular cross section. I am not aware of any previous published solution for non-rectangular sections.

The usual assumptions for elastic reinforced concrete analysis are made; i.e.:

- Plane sections remain plane.
- Tensile stresses in the concrete are ignored.
- Linear elastic stress/strain behaviour of the reinforcement and the concrete in compression.

Consider a reinforced concrete section with:

- Applied axial load,  $P$ , (compression positive) applied at height  $e$  above the compression face
- $X$  axis on the section NA,  $y$  positive towards the compression face.
- Extreme compression face at  $Y$  above the neutral axis.

The reactions in the section are:

A moment about the neutral axis,  $M$

An axial force,  $N$

For moment equilibrium:

$$M/N = Y + e$$

$$M = (Y + e)N$$

$$M - N(Y + e) = 0$$

Consider a stress distribution in moment equilibrium with the applied load, such that the stress at the compression face =  $Y$ , then:

$$N = \int y \, dA = \text{First Moment of Area about NA, } Q_x$$

$$M = \int y^2 \, dA = \text{Second Moment of Area about NA, } I_x$$

$$\text{So: } I_x - Q_x(Y + e) = 0$$

For a rectangular section:

$$\text{Width} = B$$

$$\text{Height of compression block} = Y$$

$$Q_x = BY^2/2$$

$$I_x = BY^3/3$$

$$BY^3/3 - BY^2/2(Y + e) = 0$$

$$BY^3(1/3 - 1/2) = BY^2e/2$$

$$e = (-BY^3/6) (2/BY^2) = -Y/3$$

$$Y = -3e$$

Note that since the concrete cannot take any tensile load the applied load must be below the compression face.

For a section with one layer of reinforcement:

$$\text{Transformed reinforcement area} = a_{st}$$

$$\text{Where } a_{st} = a_s E_s/E_c$$

$$E_s = \text{Young's Modulus of the reinforcing steel}$$

$$E_c = \text{Young's Modulus of the concrete}$$

$$\text{Depth below compression face} = d_{st}$$

$$Q_x = BY^2/2 + a_{st} (Y - d_{st}) \quad (1.1)$$

$$I_x = BY^3/3 + a_{st} (Y - d_{st})^2 = BY^3/3 + a_{st} (Y^2 - 2Yd_{st} + d_{st}^2) \quad (1.2)$$

$$BY^3/3 + a_{st} (Y^2 - 2Yd_{st} + d_{st}^2) - (BY^2/2 + a_{st} (Y - d_{st}))(Y + e) = 0$$

$$BY^3/3 + a_{st} (Y^2 - 2Yd_{st} + d_{st}^2) - (BY^3/2 + BY^2e/2 + a_{st} (Y^2 - Yd_{st} + Ye - d_{st}e)) = 0$$

$$-BY^3/6 - BY^2e/2 - Y(a_{st} (d_{st} - e)) + a_{st}d_{st} (d_{st} + e) = 0$$

$$B/6Y^3 + Be/2Y^2 + (d_{st} - e)a_{st} \cdot Y - d_{st} (d_{st} + e)a_{st} = 0$$

$$\text{Substituting into } aY^3 + bY^2 + cY + d = 0$$

$$a = B/6 \quad (1.3)$$

$$b = Be/2 \quad (1.4)$$

$$c = (d_{st} + e) a_{st} \quad (1.5)$$

$$d = -(d_{st} + e) d_{st} a_{st} \quad (1.6)$$

For multiple reinforcement layers:

$$Q_x = \frac{BY^2}{2} + \sum_{i=1}^{i=n} a_{sti} (Y - d_{sti}) \quad (2.1)$$

$$I_x = \frac{BY^3}{3} + \sum_{i=1}^{i=n} a_{sti} (Y^2 - 2Yd_{sti} + d_{sti}^2) \quad (2.2)$$

$$a = B/6 \quad (2.3)$$

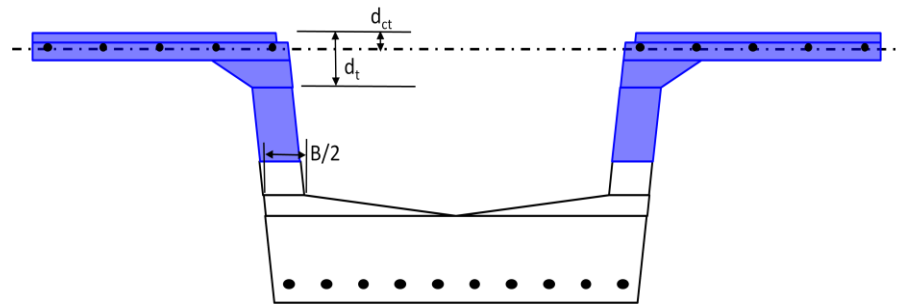
$$b = Be/2 \quad (2.4)$$

$$c = \sum_{i=1}^{i=n} (d_{sti} + e) a_{sti} \quad (2.5)$$

$$d = - \sum_{i=1}^{i=n} (d_{sti} + e) d_{sti} a_{sti} \quad (2.6)$$

Note that where  $d_{sti} < Y$  (i.e. for reinforcement layers inside the compression zone) the reinforcement is displacing an equal area of concrete, and the transformed area becomes:  
 $a_{st} = a_s (E_s/E_c - 1)$

Non rectangular concrete sections are handled by dividing the section into two layers, an upper layer of any shape, and a rectangular or trapezoidal lower, containing the neutral axis.



For the top layer:

Depth of layer =  $d_t$

Depth of layer centroid below compression face =  $d_{ct}$

Area =  $A_t$

Second moment of area about layer centroidal axis =  $I_{ct}$

Then:

First moment of area about section NA:

$$Q_{x1} = A_t(Y - d_{ct})$$

Second moment of area about section NA:

$$I_{x1} = I_{ct} + A_t(Y - d_{ct})^2 = I_{ct} + A_{tc}(Y^2 - 2Yd_{ct} + d_{ct}^2)$$

For a rectangular layer of width B containing the neutral axis:

First moment of area about section NA:

$$Q_{x2} = B(Y - d_t)^2/2 = B(Y^2 - 2d_tY + d_t^2)/2$$

Second moment of area about section NA:

$$I_{x2} = B(Y - d_t)^3/3 = B(Y^3 - 3d_tY^2 + 3d_t^2Y - d_t^3)/3$$

Total of the two layers:

$$Q_x = A_t(Y - d_{ct}) + B(Y^2 - 2d_tY + d_t^2)/2 \quad (3.1)$$

$$I_x = I_{ct} + A_{tc}(Y^2 - 2Yd_{ct} + d_{ct}^2) + B(Y^3 - 3d_tY^2 + 3d_t^2Y - d_t^3)/3 \quad (3.2)$$

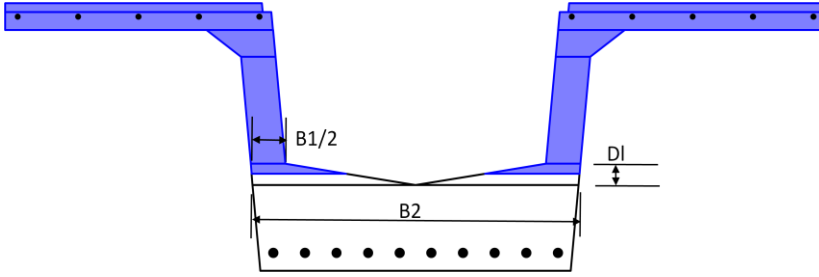
From which the following factors are derived for the neutral axis equation:

$$a = B/6 \quad (3.3)$$

$$b = Be/2 \quad (3.4)$$

$$c = (A_t d_{ct}) - e(Bd_t - A_t) - Bd_t^2 + \sum_{i=1}^{i=n} (d_{sti} + e) a_{sti} \quad (3.5)$$

$$d = -I_{ct} - (A_t d_{ct})(d_{ct} + e) + Bd_t^2 \left( \frac{d_t}{3} + \frac{e}{2} \right) - \sum_{i=1}^{i=n} (d_{sti} + e) d_{sti} a_{sti} \quad (3.6)$$



Finally for a trapezoidal lower layer the neutral axis equation becomes a quartic:

$$aY^4 + bY^3 + cY^2 + dY + e = 0$$

where:

$$K = (B2 - B1) / D1$$

$$B = B1$$

$$Q_x = A_t(Y - d_{ct}) + KY/6(Y - d_t)^2 + (3B - Kd_t)(Y - d_t)^2/6 + \sum_{i=1}^{i=n} a_{sti}(Y - d_{sti}) \quad (4.1)$$

$$I_x = I_{ct} + A_t(Y - d_{ct})^2 + KY/12(Y - d_t)^3 + (4B - Kd_t)(Y - d_t)^3/12 + \sum_{i=1}^{i=n} a_{sti}(Y^2 - 2Yd_{sti} + d_{sti}^2) \quad (4.2)$$

$$a = K/12 \quad (4.3)$$

$$b = B/6 - K/6(dt - e) \quad (4.4)$$

$$c = Be/2 - Kdte/2 \quad (4.5)$$

$$d = (A_t d_{ct}) - e(Bd_t - A_t - Kd_t^2/2) - \frac{Bd_t^2}{2} + Kd_t^3/6 + \sum_{i=1}^{i=n} (d_{sti} + e)a_{sti} \quad (4.6)$$

$$e = -I_{ct} - (A_t d_{ct})(d_{ct} + e) + Bd_t^2 \left( \frac{d_t}{3} + \frac{e}{2} \right) - \left( \frac{Kd_t^3}{6} \right) \left( \frac{d_t}{2} + e \right) - \sum_{i=1}^{i=n} (d_{sti} + e)d_{sti} a_{sti} \quad (4.7)$$

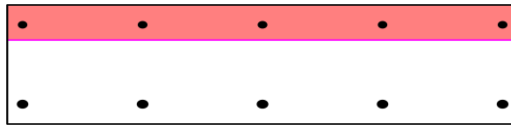
The derivation of these factors is given in Appendix A.

### Example 1:

Rectangular section: 1000 mm wide x 300 mm deep, E = 33.33 GPa

Reinforcement: Top, 5 No Y16, depth = 50 mm; Bottom, 5 No Y20, depth = 250 mm, E = 200 GPa

Axial load = 500 kN at 100 mm above the top face.



$$a = 1000/6 = 1.6667E+02 \quad (2.3)$$

$$b = 1000 * 100/2 = 5.0000E+04 \quad (2.4)$$

$$c = \sum_{i=1}^{i=n} (d_{sti} + e)a_{sti} = (50 + 100) * 1005 * 5 + (250 + 100) * 1571 * 6 = 4.0527E+06 \quad (2.5)$$

$$d = -\sum_{i=1}^{i=n} (d_{sti} + e)d_{sti} a_{sti} = -((50 + 100) * 50 * 1005 * 5 + (250 + 100) * 250 * 1571 * 6) = -8.6237E+08 \quad (2.6)$$

Solving the cubic polynomial for Y yields **Y = 88.30 mm**

$$Q_x = \frac{BY^2}{2} + \sum_{i=1}^{i=n} a_{sti}(Y - d_{sti}) \quad (2.1)$$

$$= 1000 * 88.30^2 / 2 + 1005 * 5 * (88.30 - 50) + 1571 * 6 * (88.30 - 250) = 2.567E+06$$

$$I_x = \frac{BY^3}{3} + \sum (a_{sti} (Y - d_{sti})^2) \quad (2.2)$$

$$= 1000 * 88.30^3 / 3 + 1005 * 5 * (88.30 - 50)^2 + 1571 * 6 * (88.30 - 250)^2 = 4.833E+08$$

Reaction eccentricity from top face =  $4.833+08 / 2.567E+06 - 88.30 = \mathbf{100 \text{ mm}}$

Top face stress,  $f_{cc} = MY/I_x = 1.0E+06 * (100 + 88.30) * 88.30 / 4.833E+08 = \mathbf{17.20 \text{ MPa}}$

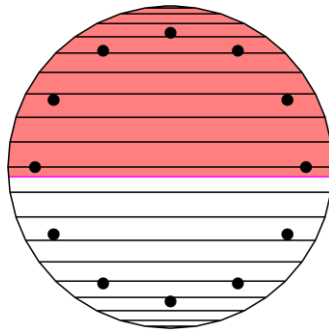
Tensile reinforcement stress =  $f_{cc} * E_s/E_c * ((Y - d_{st})/Y) = 17.20 * 6 * (88.30 - 250)/88.3 = \mathbf{-189.0 \text{ MPa}}$

### Example 2:

**Circular section:** 600mm diameter,  $E = 33.33 \text{ GPa}$

Reinforcement: 12 No. Y 20 bars, cover = 40 mm,  $E = 200 \text{ GPa}$

Axial load = 1000 kN at 100 mm below the outer compression fibre.



By dividing the cross section into layers it is found that the neutral axis is between 300 and 350 below the compression fibre; therefore divide the circular section into two parts:

A semi-circle of radius 300 mm.

A trapezoidal layer:  $B1 = 600 \text{ mm}$ ,  $B2 = 592.6 \text{ mm}$ ,  $DI = 46.93 \text{ mm}$ .

$$a = -1.312E-02 \quad (4.3)$$

$$b = 1.105E+02 \quad (4.4)$$

$$c = -3.236E+04 \quad (4.5)$$

$$d = 5.643E+06 \quad (4.6)$$

$$e = -1.942E+09 \quad (4.7)$$

Solving the quartic polynomial for  $Y$  yields  $\mathbf{Y = 318.05 \text{ mm}}$

$$Q_x = 2.061E+07 \text{ mm}^3 \quad (4.1)$$

$$I_x = 4.494E+09 \text{ mm}^4 \quad (4.2)$$

Reaction eccentricity from top face =  $4.494E+09 / 2.061E+07 - 318.05 = \mathbf{-100 \text{ mm}}$

Top face stress,  $f_{cc} = MY/I_x = 1.0E+06 * (-100 + 318.05) * 318.05 / 4.494E+09 = \mathbf{15.4 \text{ MPa}}$

Max tensile reinforcement stress =  $f_{cc} * E_s/E_c * ((Y - d_{st})/Y)$   
 $= 15.4 * 6 * (318.05 - 550)/318.05 = \mathbf{-67.5 \text{ MP}}$

**Reference:**

- (1) Ghali, A, Favre, R. and Elbadry, M (2002), *Concrete Structures, Stresses and Deformations*, Spon Press, London, Appendix D

**Appendix A**

**Derivation of Neutral Axis Equation Factors for a Trapezoidal Lower Layer:**

Trapezoid side slope,  $K = (B_2 - B_1)/D_t$

$B = B_1$

**Layer containing NA**

$$Q_x = K/6 * (Y^3 - 2Y^2d_t + Yd_t^2) + (3B - Kd_t)(Y^2 - 2Yd_t + dt^2)/6$$

$$I_x = K/12 * (Y^4 - 3d_tY^3 + 3d_t^2Y^2 - d_t^3Y) + (4B - Kd_t)(Y^3 - 3d_tY^2 + 3d_t^2Y - d_t^3)/12$$

**Total (Concrete Only)**

$$Q_x = A_t(Y - d_c) + K/6 * (Y^3 - 2Y^2d_t + Yd_t^2) + (3B - Kd_t)(Y^2 - 2Yd_t + dt^2)/6$$

$$-Q_x(Y+e) = -A_t(Y^2 - d_cY) - K/6(Y^4 - 2Y^3*d_t + Y^2d_t^2) - (3B - Kd_t)(Y^3 - 2Y^2d_t + d_t^2Y)/6 - A_t e(Y - d_c) - Ke/6 * (Y^3 - 2Y^2d_t + Yd_t^2) - (3B - Kd_t)e(Y^2 - 2Yd_t + d_t^2)/6$$

$$I_x = I_{ct} + A_t(Y^2 - 2Yd_c + d_c^2) + K/12(Y^4 - 3d_tY^3 + 3d_t^2Y^2 - d_t^3Y) + (4B - Kd_t)(Y^3 - 3d_tY^2 + 3d_t^2Y - d_t^3)/12$$

	$Y^4$	$Y^3$	$Y^2$	$Y$	$1$
$I_{xc}$	$-K/12$	$-B/3 + Kd_t/3$	$-A_t - Kd_t^2/2 + Bd_t$	$2A_t d_c + Kd_t/12 - (4B - Kd_t)d_t^2/4$	$-I_{ct} - A_t d_c^2 + b d_t^3/3 - Kd_t^4/12$
$-Q_{xc}(Y+e)$	$K/6$	$B/2 - K/6(3d_t - e)$	$A_t - B(d_t - e/2) + Kd_t/2(d_t - e)$	$-A_t(d_c - e) + Bd_t(d_t/2 - e) - Kd_t^2/2(d_t/3 - e)$	$-e(A_t d_c - Bd_t^2/2 + Kd_t^3/6)$
Total	$K/12$	$B/6 - K/6(d_t - e)$	$Be/2 - Kd_t e/2$	$-Bd_t^2/2 + Kd_t^3/6 + A_t d_c - e(Bd_t - A_t - Kd_t^2/2)$	$-I_{ct} - A_t d_c(d_c + e) + Bd_t^2(d_t/3 + e/2) - Kd_t^3/6(d_t/2 + e)$

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